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Force in static general relativity

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Abstract. The force acting on the matter inside an arbitrary closed surface S is considered for the case of a static space–time $g_{\alpha\beta}$ with a static matter tensor $T^{\alpha\beta}$ satisfying $T^{\alpha\beta}{}_{;\beta}=0$. Since the matter is in equilibrium, the body force is balanced by the surface force. The component of this force in any chosen direction is defined as the surface integral of an invariant and is transformed into a volume integral. Shrinking S to a point then yields the density of the body force. This paper is mostly concerned with applying this result to special situations. For instance, it is found that in the presence of shear stress the body force can have a non-zero component perpendicular to the gradient of g_{44}.

1. Introduction

We restrict our attention in this paper to space–times $g_{\alpha\beta}$ which are static and possess a static matter tensor $T^{\alpha\beta}$. We use coordinates such that

$$g_{\alpha\beta,4}=g_{i4}=T^{\alpha\beta}{}_{,4}=T^{i4}=0, \tag{1}$$

where commas indicate partial derivatives, the Greek indices range from 1 to 4, the Latin indices range from 1 to 3, $x_4=t$, and the signature of $g_{\alpha\beta}$ is +2. We do not insist that the Einstein field equations are satsified and we therefore include the case of test bodies, i.e. bodies with $T^{\alpha\beta}$ so small that its influence on the metric is negligible.

The closely related concepts of inertial mass, active and passive gravitational mass (Will 1976, Rohrlich 1965) and force, together with their corresponding densities, are useful in both finding and interpreting a $g_{\alpha\beta}$ or a $T^{\alpha\beta}$. The present work is mostly concerned with force. In § 2 we consider a portion of a body at rest. Since the surface force is then in balance with the body force, the former is used to obtain an expression for the density of the latter (equation (10)). Section 3 gives a comparison of our results with some results from the literature. The implications of (10) are examined in the remaining parts of the paper. In § 4 we first calculate the component of the body force in a direction which is an eigenvector of $T^{\alpha\beta}$ (§ 6 and most of § 5 are examples of this and § 7 suggests that this case is consistent with special relativity). A second case shows that in the presence of shear stresses the body force can have a non-zero component orthogonal to $g_{44,a}$. This result suggested part (iv) of § 5 and thus demonstrates the usefulness of the concept of body force in general relativity.

2. From the surface force to the body force density

Covariant differentiation in the static space V_4 with metric $g_{\alpha\beta}$ is denoted by a semicolon; in the space V_3 with the equation $x_4=$ constant and metric g_{ik} it is denoted

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by a vertical stroke. The determinants of the respective metric tensors are ${}^3g$ and ${}^4g$. The 27 Christoffel symbols Γ^a_{bc} are the same for both spaces. In V_4 it is assumed that there is a static matter tensor which satisfies the equations of motion

$$T^{\alpha\beta}{}_{;\beta} = 0. \tag{2}$$

Since $T^{\alpha\beta}$ is static, the matter is in equilibrium. Therefore the body force F^a acting on the matter inside a region V of V_3 is balanced by the surface force acting on the surface S of V. The surface force acting on an element $\mathrm{d}S$ of S is

$$T^{\alpha\beta} n_b \, \mathrm{d}S, \tag{3}$$

where the 3-vector n_b is the outward unit normal to S. Let us choose a direction by picking a unit 3-vector $u_a(O)$ defined at a point O in V. In order to sum the components of these surface forces in this chosen direction, we move u_a by parallel transport along the unique geodesic from O to the general point P on S. The component of the surface force at P in the direction u_a is then

$$T^{ab} n_b u_a \, \mathrm{d}S. \tag{4}$$

Note that a vector undergoes parallel transport along a curve if its absolute derivative vanishes (Synge 1964, p 12). The scalar product is invariant under parallel transport, so if instead of u_a the vector $T^{ab}n_b$ were moved by parallel transport from P to O, the result would be the same. Integrating over S, we define the component of the force F^a in the direction $u_a(O)$ by

$$F^a u_a(O) \equiv \int_S T^{ab} n_b u_a \, \mathrm{d}S. \tag{5}$$

Gauss's theorem then yields

$$F^a u_a(O) = \int_V (T^{ab} u_a)_{|b} \, \mathrm{d}V = \int_V (T^{ab}{}_{|b} u_a + T^{ab} u_{a|b}) \, \mathrm{d}V, \tag{6}$$

where $\mathrm{d}V = ({}^3g)^{1/2} \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3$. The term $u_{a|b}$ is exactly zero if the Riemann tensor of V_3 is zero, as in § 5. This term goes to zero as V shrinks to a point, as can be seen from Synge (1964, equation (71), p 59 and the 2nd or 3rd of equation (111), p 67). The component of the body force f^a in the arbitrary direction u_a is thus defined by

$$f^a u_a \equiv \lim_{V \to 0} \frac{1}{V} F^a u_a(O) = T^{ab}{}_{|b} u_a, \tag{7}$$

where V is the volume of the region V.

We have therefore, because of the arbitrariness of u_a, that

$$f^a = T^{ab}{}_{|b}. \tag{8}$$

Since $\Gamma^a_{4b} = T^{a4} = 0$, we have

$$T^{ab}{}_{|b} = T^{a\beta}{}_{;\beta} - T^{a4}{}_{;4} = -T^{a4}{}_{;4} = -\Gamma^a_{44} T^{44} - \Gamma^4_{4b} T^{ab}, \tag{9}$$

and (7) and (8) become

$$f^a u_a = \tfrac{1}{2}(g^{ab} g_{44,b} T^{44} - g^{44} g_{44,b} T^{ab}) u_a, \tag{10a}$$

$$f^a = \tfrac{1}{2} g^{44} g_{44,b} (g^{ab} T^4_4 - T^{ab}). \tag{10b}$$

Some justification for using parallel transport in (4) is given by the following *theorem*. If in a static space–time with $g_{\alpha\beta,4} = g_{a4} = 0$, a matter tensor $T^{\alpha\beta}$ satisfying (2) has only T^{11} non-zero, then the x_1 lines are geodesics in V_4 and V_3. In § 5, where the background metric has a flat V_3, parallel transport needs no justification since there its use coincides with standard procedures. Oliver (1977), in defining a matter tensor for extended bodies, uses parallel transport of tensors along geodesics. Use of parallel transport of a vector as in (6) resembles that of Temple (1936).

3. Comparison with the literature

Whittaker (1935) and Ruse (1935) take the gravitational force to be proportional to the acceleration:

$$h^\alpha \equiv \frac{\delta}{\delta s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\frac{\mathrm{d}x^\gamma}{\mathrm{d}s}, \tag{11}$$

where s is the proper time. They take $\mathrm{d}x^i/\mathrm{d}s = 0$, thus assuming (see § 7) that matter at rest is composed of particles at rest, and find that

$$h^4 = 0, \qquad h^a = \tfrac{1}{2}g^{44}g_{44,b}g^{ab}. \tag{12}$$

Eddington (1924) writes (2) in the form

$$k_\alpha \equiv (T^\beta_\alpha|{}^4g|^{1/2})_{,\beta} = \tfrac{1}{2}g_{\beta\delta,\alpha}|{}^4g|^{1/2}T^{\beta\delta} \tag{13}$$

and takes this 'as the (negative) body-force acting on unit-volume'.

Synge (1964, p 249) discusses, without endorsement,

$$\int (T^{\alpha\beta}|{}^4g|^{1/2})_{,\beta}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$$

as the total force acting on a body, which with (2) leads to the body force density

$$l^\alpha \equiv (T^{\alpha\beta}|{}^4g|^{1/2})_{,\beta} = -\Gamma^\alpha_{\beta\delta}|{}^4g|^{1/2}T^{\beta\delta}. \tag{14}$$

To compare the expressions (10), (13), (14) for the case of a static metric and static matter we assume (1) and $k^\alpha = g^{\alpha\beta}k_\beta$ and we find for the body force densities

$$f^4 = k^4 = l^4 = 0,$$

$$|{}^3g|^{1/2}f^a = |{}^3g|^{1/2}(-\Gamma^a_{44}T^{44} - \Gamma^4_{4b}T^{ab}), \tag{15}$$

$$k^a = |{}^4g|^{1/2}(-\Gamma^a_{44}T^{44} + \tfrac{1}{2}g_{bc,d}g^{ad}T^{bc}), \tag{16}$$

$$l^a = |{}^4g|^{1/2}(-\Gamma^a_{44}T^{44} - \Gamma^a_{bc}T^{bc}). \tag{17}$$

Using a divergence as body force density in (13) and (14) enables one to use Green's theorem to establish many parallels between general relativity and Newtonian mechanics, but at the cost of losing tensorial invariance. Thus Synge, while pointing out these parallels, does not claim physical significance for them.

The approach of § 2 evolved from doing simple examples as in §§ 5 and 6 where it was obvious what the surface force and body force ought to be. It is therefore not surprising that f^a, but not k^a or l^a, leads in § 5 to the expected answers. From the tensor character of $T^{\alpha\beta}$ and (8) it follows that f^a is a vector under transformation of the space-like variables.

For later comparison we conclude this section by quoting some results on gravitational mass since, in the static case, gravitational mass is measured indirectly by measuring gravitational force (= body force). Making no distinction between active and passive gravitational mass, Synge (1937), Pirani (1956) and Møller (1962) conclude that the density of gravitational mass is dependent on $T^{\alpha\beta}$ via

$$-T^4_4 + T^a_a = -2T^4_4 + T^\beta_\beta. \tag{18}$$

Indeed, in Einstein's linearised theory expression (18) is proportional to the D'Alembertian of g_{44} and thus plays the role of active gravitational mass. Tolmann (1934), Møller (1962), Whittaker (1935), Ruse (1935) and many others find that the total gravitational mass of an isolated body is

$$\int_{\text{Body}} (-T^4_4 + T^a_a)(-g_{44})^{1/2}\,\mathrm{d}V. \tag{19}$$

Some proofs use pseudo-tensors, and all of them use Einstein's equations; Whittaker (1935) and Synge (1937) start from the density of some gravitational force. For a spherically symmetric body, (19) is indeed equal to the constant m appearing in the exterior Schwarzschild metric. In (19), the contribution of the stresses, i.e. T^a_a, is usually small even when T^a_a is of the same order as T^4_4 (as is the case for electromagnetic fields for which $T^\beta_\beta = 0$), because (Misner and Putnam 1959), for a bounded isolated body in almost flat space, the integral over T^a_a is almost exactly zero. For example, electromagnetic radiation in a box produces tension in the walls of the box which almost cancels the contributions of the Maxwell stresses to (19).

4. Two cases

To gain insight into (10) we study two special cases.

4.1. Case A

Denote the eigenvalues and eigenvectors of T^{ab} by $\theta_{(c)}$ and $\lambda^{\;b}_{(c)}$ respectively. If u_b, the direction in which we find the component of the force f^a, coincides with one eigenvector, say $\lambda^{\;b}_{(1)}$, then with

$$T^{ab}u_a = \theta_{(1)}u^b, \tag{20}$$

we find from (10a)

$$f^a u_a = \tfrac{1}{2}u^b g_{44,b}\,g^{44}(T^4_4 - \theta_{(1)}). \tag{21}$$

If furthermore u_b is a positive multiple of $g_{44,b}$, then

$$f^a u_a = \tfrac{1}{2}\|g_{44,b}\|g^{44}(T^4_4 - \theta_{(1)}), \tag{22}$$

as demonstrated by examples in the next two sections. Gravitational force measures passive gravitational mass which, for the special situation considered here, is equal to

$$-T^4_4 + \theta_{(1)}. \tag{23}$$

We recall that Synge (1937), Pirani (1956) and Møller (1962) obtained (18)

$$-T^4_4+\theta_{(1)}+\theta_{(2)}+\theta_{(3)} \tag{24}$$

for the density of the (active) gravitational mass.

4.2. *Case B*

We now investigate the component of f^a perpendicular to $g_{44,a}$ and thus assume

$$u^a g_{44,a}\equiv g^{ab}g_{44,b}u_a=0. \tag{25}$$

Then only the second term in (10*a*) remains and we have

$$f^a u_a=-\tfrac{1}{2}g^{44}g_{44,b}T^b_a u^a. \tag{26}$$

This can be non-zero, i.e. the component of the force f^a orthogonal to $g_{44,a}$ need not vanish. If we choose the coordinates x_a such that $g_{44,a}$ is tangent to the x_1 lines, then

$$u^1=g_{44,2}=g_{44,3}=0 \tag{27}$$

and thus

$$f^a u_a=-\tfrac{1}{2}g^{44}g_{44,1}(T^1_2u^2+T^1_3u^3). \tag{28}$$

Part (iv) of § 5 gives an example of this force orthogonal to $g_{44,a}$.

5. Simplest example

We apply here the results of §§ 2, 3, 4 to the simplest non-trivial $g_{\alpha\beta}$. Suppose the metric is given by $(x_1\, x_2\, x_3\, x_4\equiv x\, y\, z\, t)$

$$\mathrm{d}s^2=\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2-W^2(x)\,\mathrm{d}t^2, \tag{29}$$

which is vacuum if $-g_{44}\equiv W^2=x^2$. The 3-space is flat, and thus, for the vector u_a undergoing parallel transport, we have

$$u_{a|b}=0 \text{ everywhere.} \tag{30}$$

The non-zero Γ's are

$$\Gamma^4_{41}=W_{,1}/W, \qquad \Gamma^1_{44}=WW_{,1}, \tag{31}$$

and material particles, initially at rest, fall along x lines. We assume for simplicity that the non-zero $T^{\alpha\beta}$'s are at most

$$T^{11},\ T^{12},\ T^{22},\ T^{44}, \tag{32}$$

and are functions of x and y only. The non-trivial equations (2) then become

$$(WT^{11})_{,1}+(WT^{12})_{,2}+W_{,1}T^4_4=0, \tag{33a}$$

$$(WT^{21})_{,1}+(WT^{22})_{,2}=0. \tag{33b}$$

If in addition

$$T^{12}=T^{22}=0 \tag{34}$$

is assumed, then any cylinder with equation

$$H(y,z)=0 \tag{35}$$

can be considered as a free surface of a body, because for this surface

$$T^{ab}n_b = 0. \tag{36}$$

(i) If we assume

$$T^{12} = T^{22} = T^1_1 - T^4_4 = 0, \tag{37a}$$

then from (33a) we obtain

$$T^1_1 = T^4_4 = \text{arbitrary function of } y. \tag{37b}$$

The force $\int T^{11}n_1u_1\,\mathrm{d}S$ on the part of the surface $x = a$ inside the cylinder $H(y, z) = 0$ is therefore the same as that for the surface $x = b$ (if $u_a = n_a = (1, 0, 0)$). In calculating the sum of all surface forces acting on the body between these two planes, we have the two n_a anti-parallel (n_a is the outward normal) and the sum of all surface forces is therefore zero; i.e. matter between the two surfaces has, as required by (22), no weight. If instead of f^a we would use the expressions (16), (17) for the body force, we find for the example considered here

$$k^1 = l^1 = W(-\Gamma^1_{44}T^{44}) = -W_{,1}T^{44}, \qquad k^2 = l^2 = k^3 = l^3 = 0, \tag{38}$$

and the volume integral $\int l^a u_a\,\mathrm{d}V$ is not zero, i.e. is not equal to the corresponding integral over surface forces. Equation (18) for the $T^{\alpha\beta}$ of (32) and (37) predicts a vanishing density of gravitational mass which is consistent with a vanishing integral over surface forces; but (18) is eliminated from the competition by the next case.

(ii) Adding to any solution of (33) the solution

$$T^{22} = \text{arbitrary function of } x \tag{39}$$

gives another solution which for any closed S has the same total surface force as the initial solution; this fact for the case $T^{12} = 0$ demonstrates that $T^{22} = \theta_{(2)}$ does not enter (21). The fact that T^{22} does not affect the weight of the matter inside S is equivalent to the fact that for test particles, $x(t)$ is independent of $\dot{y}(0)$, the initial velocity in the y direction (see § 7). In this case, (18) gives a density of the gravitational mass proportional to T^2_2 and this result is not consistent with the vanishing of the integral over the surface forces.

(iii) If a solution of (33) satisfies

$$T^{12} = T^{22} = T^{11}(x = a) = 0, \qquad T^{11}(x = b) = C, \tag{40}$$

for some constants a, b, C, then the substitution

$$T^{11} \to T^{11} + \text{constant}/W \tag{41}$$

leaves T^4_4 unchanged, and we can find some constant D such that

$$T^{12} = T^{22} = T^{11}(x = b) = 0, \qquad T^{11}(x = a) = D. \tag{42}$$

For example, for $T^4_4 = \theta = \text{constant}$, the transition (41) from (40) to (42) is given by

$$T^{11} = \theta[(W(a)/W(x)) - 1] \to T^{11} = \theta[(W(b)/W(x)) - 1] \tag{43}$$

(smooth time-dependent transitions with $T^{i4} = 0$ are easy to find). We interpret two such solutions as belonging to the same body; first, (40), supported at $x = b$ and free at $x = a$, then, (42), supported at $x = a$ and free at $x = b$. We have

$$D \neq -C, \tag{44}$$

and thus agree with Nordtvedt (1975) 'that the weight of a body must necessarily depend on where the supporting force is located'. For an opposing view see Grøn (1977, 1979).

(iv) To demonstrate the existence of the force orthogonal to $g_{44,b}$, produced by shear stress, let us investigate matter in a box bounded by the 'planes'

$$x = a \pm o, \qquad y = \pm p, \qquad z = 0, \qquad z = 1. \tag{45}$$

Equations (33) have the solution ($\alpha(x), \beta(y), \delta(y)$ are arbitrary functions of their arguments)

$$WT^{22} = \beta\alpha_{,11}, \qquad WT^{21} = \delta - \alpha_{,1}\beta_{,2},$$
$$WT^{11} = \alpha\beta_{,22} - x\delta_{,2} - W\int W_{,1}T^4_4\,\mathrm{d}x. \tag{46}$$

We demand that on the body's boundary only vertical surface forces are acting (i.e. forces parallel to $g_{44,b}$); we therefore impose the conditions

$$T^{22} = 0 \text{ at } y = \pm p, \qquad T^{12} = 0 \text{ at } x = a \pm o \text{ and at } y = \pm p. \tag{47}$$

Newtonian mechanics would then predict that the total force transmitted across any vertical cross-section of the body is zero; § 4.2 suggests this is not so in general relativity. Indeed, taking in (46)

$$\delta = 0, \qquad \beta = (y^2 - p^2)^2, \qquad \alpha_{,1} = (x-a)^2 - o^2, \tag{48}$$

as the solution of (47), we find

$$WT^{22}(x, 0) = 2(x-a)p^4, \tag{49}$$

and the (horizontal) force K across the section $y = 0$,

$$K \equiv \int_{x=a-o}^{a+o} T^{22}(x, 0)\,\mathrm{d}x, \tag{50}$$

is generally non-zero. For $W = x$ we obtain for instance for T^{22} of (49)

$$K = 2p^4\{2o + a\ln[(a-o)/(a+o)]\}. \tag{51}$$

The K of (50) agrees of course with the integral of the body force over the part of the body on one side of the section $y = 0$:

$$K = \int_{y=-p}^{0}\int_{x=a-o}^{a+o} f^2(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{52}$$

The formulae (16) and (17) in this case give $k^2 = l^2 = 0$, i.e. they predict a vanishing force across vertical cross sections.

6. Another example

To illustrate some subtleties, we consider here the vacuum metric (a Kasner metric)

$$\mathrm{d}s^2 = x^4(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) - x^{-2}\,\mathrm{d}t^2. \tag{53}$$

The non-zero Γ's are

$$\Gamma^1_{44} = -x^{-7}, \qquad \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = -2\Gamma^4_{14} = -\Gamma^1_{22} = -\Gamma^1_{33} = 2/x. \tag{54}$$

(i) We first consider solutions of (2) with only T^1_1 and T^4_4 non-zero, and both functions of x only. The only non-trivial equation of (2) is then

$$T^k_{1|k}=0=x^{-3}(T^1_1x^3)_{,1}-x^{-1}T^4_4, \tag{55}$$

and any surface given by (35) can be considered as the free surface of a body. If in addition we demand that the force on the surface $x=a$ is the same as that on $x=b$ for all a and b, then with

$$u_c=\pm n_c=(g_{11}{}^{1/2},0,0), \tag{56}$$

we find from

$$T^{11}u_1n_1\,\mathrm{d}S=(T^1_1g^{11})(g_{11}{}^{1/2})(g_{11}{}^{1/2})(\mathrm{d}y\,g_{22}{}^{1/2}\,\mathrm{d}z\,g_{33}{}^{1/2})=\text{constant}, \tag{57}$$

and with (55), that

$$T^1_1=T^4_4=\text{constant}\,x^{-4}. \tag{58}$$

Since the sum of all surface forces vanishes, we find as required by (21) that

$$T^4_4-T^1_1=0. \tag{59}$$

(ii) The solution of (2) with only T^2_2 and T^4_4 non-zero and functions of x is given by

$$2T^2_2(x)=T^4_4(x). \tag{60}$$

Since $T_1{}^kn_k=0$ for any surface $x=$ constant, such matter might seem to stay at rest without support and thus seems to violate conclusions reached from (21). But two planes with equations $y=c$ and $y=d$ respectively are not parallel since their true distance is $g_{22}{}^{1/2}|d-c|$; also the y lines are not space-like geodesics. The equilibrium of the matter (60) resembles therefore that of an arch in Newtonian statics, except that in general relativity the arch can have infinite length.

We now calculate the net surface force produced by T^2_2 on a volume element $\mathrm{d}V$ bounded by

$$\begin{aligned} &x=X, && y=Y, && z=Z,\\ &x=X+\mathrm{d}X, && y=Y+\mathrm{d}Y, && z=Z+\mathrm{d}Z. \end{aligned} \tag{61}$$

Taking

$$U_a\equiv u_a(X,Y,Z)=u_a(O)=(g_{11}^{1/2}(X),0,0), \tag{62}$$

we have for the face $y=Y$

$$u_a=(g_{11}^{1/2}(x),0,0),\qquad n_a=(0,-g_{22}^{1/2}(x),0), \tag{63}$$

and the surface force (4) is

$$T^{22}n_2u_2\,\mathrm{d}S=0. \tag{64}$$

For the face $y=Y+\mathrm{d}Y$ we use the law of parallel transport and (62) to obtain

$$u_2(x,Y+\mathrm{d}Y,z)=U_2+\mathrm{d}u_2=0+\Gamma^a_{2b}U_a\,\mathrm{d}x^b=\Gamma^1_{22}U_1\,\mathrm{d}Y=-2X\,\mathrm{d}Y, \tag{65}$$

and we find with

$$n_a=(0,+g_{22}{}^{1/2}(x),0) \tag{66}$$

and (60) that the surface force (4) is

$$T^{22}n_2u_2\,\mathrm{d}S=(T^2_2g^{22})(g_{22}{}^{1/2})(-2X\,\mathrm{d}Y)(\mathrm{d}X\,g_{11}{}^{1/2}\,\mathrm{d}Z\,g_{33}{}^{1/2})=-T^4_4X^3\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z. \tag{67}$$

This is the total surface force in the direction u_a, since there are no forces on the remaining four faces; (67) is indeed equal to the body force, since using (21), (53) and (62) we find

$$f^au_a\,\mathrm{d}V=\tfrac{1}{2}g^{44}g_{44,1}(T^4_4-T^1_1)|^3g|^{1/2}u_1\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z=-T^4_4X^3\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z. \tag{68}$$

7. A link with special relativity

All examples in §§ 5 and 6, with the exception of (iv) in § 5, illustrate case A of § 4, i.e. the gravitational force in the direction of $g_{44,a}$ depends on the principal stress whose eigenvector is parallel to $g_{44,a}$, but does not depend on principal stresses whose eigenvectors are orthogonal to $g_{44,a}$.

Since gravitational force measures gravitational mass which presumably equals inertial mass, we expect in special relativity that the density of inertial mass depends on the orientation of the principal directions with respect to the direction of acceleration. Juxtaposition of the following two points makes this plausible.

I. Stresses are the macroscopic manifestation of microscopic motion. This is illustrated by three examples.

(i) In a perfect fluid at rest, the pressure p measures the kinetic energy of the molecules: $p=\frac{1}{3}\rho\overline{v^2}$.

(ii) Synge (1972, p 209) models compressive/tensile stress as exchange of particles with positive/negative mass.

(iii) Quantum theory models forces as the exchange of (possibly virtual) particles: electromagnetic forces–photons, molecular forces–electrons, nuclear forces–pions, etc.

II. Weizel (1955) shows as follows that in *special relativity* the inertial mass of a moving particle is a tensor (or see Silberstein (1924)). Let m, s, $x_\alpha=x_\alpha(s)$, T_α be the rest mass, proper time, world line and 4-force of a particle. The equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(m\frac{\mathrm{d}x_\alpha}{\mathrm{d}s}\right)=T_\alpha \tag{69}$$

give, for $\alpha=i$, with the abbreviations

$$v_i\equiv\mathrm{d}x_i/\mathrm{d}t,\qquad v\equiv(v_iv_i)^{1/2},\qquad \gamma\equiv\mathrm{d}t/\mathrm{d}s=(1-v^2)^{-1/2},$$
$$w_i\equiv v_i/v,\qquad P_i\equiv T_i/\gamma, \tag{70}$$

that

$$P_i=\frac{\mathrm{d}}{\mathrm{d}t}(v_i\gamma m)=\frac{\mathrm{d}}{\mathrm{d}t}(w_iv\gamma m)=w_im\gamma^3\frac{\mathrm{d}v}{\mathrm{d}t}+m\gamma v\frac{\mathrm{d}w_i}{\mathrm{d}t}. \tag{71}$$

The first term is the component of the relative 3-force P_i in the direction of v_i, and the second term is the component perpendicular to it. Comparison with the Newtonian analogue of (71) shows that acceleration perpendicular to v_i is resisted by the transversal mass

$$m_\mathrm{t}\equiv m\gamma \tag{72a}$$

and acceleration in the direction of v_i is resisted by the longitudinal mass

$$m_l \equiv m\gamma^3. \tag{72b}$$

Thus, in special relativity the inertial mass of a moving particle is a tensor and we have

$$m_l/m_t = \gamma^2 = (1-v^2)^{-1} = 1 + v^2 - v^4 + \dots. \tag{73}$$

To summarise I and II: since stress is a macroscopic manifestation of microscopic motion of particles, and since motion makes inertial mass a tensor, we expect in special relativity that the density of the inertial mass of stressed matter is a tensor.

But our aim is to discover the degree of agreement between (73) and some analogue of it in general relativity. We consider static matter which is the superposition of steady streams of incoherent dust so that the microscopic motion can be taken to be known. Restricting ourselves to metrics such as (29) or (53), we can assume that the velocity in the z direction is zero. We compare (73) with the ratio of the body forces for two such static matter tensors, both having the same T^4_4, and both constructed from particles with the same speed v ($v^2 = v_a v^a$) but the first matter tensor corresponding to particles with velocities along x lines, and the second matter tensor with velocities along y lines (all this can be achieved for any particular x). Using

$$u_a = (g_{11}{}^{1/2}, 0, 0), \tag{74}$$

we find from (21) and from the matter tensor for incoherent dust the required ratio

$$\begin{aligned}\frac{(f^a u_a)\text{1st}}{(f^a u_a)\text{2nd}} &= \frac{T^4_4 - T^1_1}{T^4_4} = 1 - \frac{T^1_1}{T^4_4}\\ &= 1 - \theta \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}x}{\mathrm{d}s} g_{11} \Big/ \theta \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s} g_{44}\\ &= 1 - (\mathrm{d}x/\mathrm{d}t)^2 g_{11} g^{44} = 1 - v^2 g_{44}.\end{aligned} \tag{75}$$

This ratio of forces (i.e. physical components) is equal to the ratio of the gravitational masses and we have good agreement with (73) if $v \ll 1$ and $g^{44} \approx -1$. We note that the first and last expression in (75) are manifestly invariants under transformations of the space-like coordinates.

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